Transfer Functions for Uniform Homogeneous Flow

As an example of a multiphase flow that exhibits the solution structure described in section (Nrl), we shall explore the form of the solution for the inviscid, frictionless flow of a two component, gas and liquid mixture in a straight, uniform pipe. The relative motion between the two components is neglected so there is only one velocity, $u(s,t)$. Surface tension is also neglected so there is only one pressure, $p(s,t)$. Moreover, the liquid is assumed incompressible (ρ_L constant) and the gas is assumed to behave barotropically with $p \propto \rho_G^k$. Then the three equations governing the flow are the continuity equations for the liquid and for the gas and the momentum equation for the mixture which are, respectively

$$\frac{\partial}{\partial t}(1-\alpha) + \frac{\partial}{\partial s}[(1-\alpha)u] = 0 \quad \text{(Nrm1)}$$

$$\frac{\partial}{\partial t}(\rho_G \alpha) + \frac{\partial}{\partial s}(\rho_G \alpha u) = 0 \quad \text{(Nrm2)}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s}\right) = -\frac{\partial p}{\partial s} \quad \text{(Nrm3)}$$

where ρ is the usual mixture density. Note that this is a system of order $N = 3$ and the most convenient flow variables are p, u and α. These relations yield the following equations for the perturbations:

$$-i\omega\tilde{\alpha} + \frac{\partial}{\partial s}\left[(1-\bar{\alpha})\tilde{u} - \bar{u}\tilde{\alpha}\right] = 0 \quad \text{(Nrm4)}$$

$$i\omega\bar{\rho}_G\tilde{\alpha} + i\omega\bar{\alpha}\tilde{\rho}_G + \bar{\rho}_G\bar{\alpha}\frac{\partial \tilde{u}}{\partial s} + \bar{\rho}_G\bar{u}\frac{\partial \tilde{\alpha}}{\partial s} + \bar{\alpha}\bar{u}\frac{\partial \tilde{\rho}_G}{\partial s} = 0 \quad \text{(Nrm5)}$$

$$-\frac{\partial \tilde{p}}{\partial s} = \bar{\rho}\left[i\omega\tilde{u} + \bar{u}\frac{\partial \tilde{u}}{\partial s}\right] \quad \text{(Nrm6)}$$

where $\tilde{\rho}_G = \tilde{p}\bar{\rho}_G/k\bar{p}$. Assuming the solution has the simple form

$$\left\{\begin{array}{c} \tilde{p} \\ \tilde{u} \\ \tilde{\alpha} \end{array}\right\} = \left\{\begin{array}{c} P_1 e^{i\kappa_1 s} + P_2 e^{i\kappa_2 s} + P_3 e^{i\kappa_3 s} \\ U_1 e^{i\kappa_1 s} + U_2 e^{i\kappa_2 s} + U_3 e^{i\kappa_3 s} \\ A_1 e^{i\kappa_1 s} + A_2 e^{i\kappa_2 s} + A_3 e^{i\kappa_3 s} \end{array}\right\} \quad \text{(Nrm7)}$$

it follows from equations (Nrm4), (Nrm5) and (Nrm6) that

$$\kappa_n(1-\bar{\alpha})U_n = (\omega + \kappa_n\bar{u})A_n \quad \text{(Nrm8)}$$

$$(\omega + \kappa_n\bar{u})A_n + \frac{\bar{\alpha}}{k\bar{p}}(\omega + \kappa_n\bar{u})P_n + \bar{\alpha}\kappa_n U_n = 0 \quad \text{(Nrm9)}$$

$$\bar{\rho}(\omega + \kappa_n\bar{u})U_n + \kappa_n P_n = 0 \quad \text{(Nrm10)}$$

Eliminating A_n, U_n and P_n leads to the dispersion relation

$$(\omega + \kappa_n\bar{u})\left[1 - \frac{\bar{\alpha}\bar{\rho}}{k\bar{p}}\frac{(\omega + \kappa_n\bar{u})^2}{\kappa_n^2}\right] = 0 \quad \text{(Nrm11)}$$

The solutions to this dispersion relation yield the following wavenumbers and velocities, $c_n = -\omega/\kappa_n$, for the perturbations:

- $\kappa_1 = -\omega/\bar{u}$ which has a wave velocity, $c_0 = \bar{u}$. This is a purely kinematic wave, a concentration wave that from equations (Nrm8) and (Nrm10) has $U_1 = 0$ and $P_1 = 0$ so that there are no pressure or velocity fluctuations associated with this type of wave. In other, more complex flows, kinematic waves may have some small pressure and velocity perturbations associated with them and their velocity may not exactly correspond with the mixture velocity but they are still called kinematic waves if the major feature is the concentration perturbation.
- $\kappa_2, \kappa_3 = -\omega/(\bar{u} \pm c)$ where c is the sonic speed in the mixture, namely $c = (k\bar{p}/\bar{\alpha}\bar{\rho})^{\frac{1}{2}}$. Consequently, these two modes have wave speeds $c_2, c_3 = \bar{u} \pm c$ and are the two acoustic waves traveling downstream and upstream respectively.

Finally, we list the solution in terms of three unknown, complex constants P_2, P_3 and A_1:

$$\left\{ \begin{array}{c} \tilde{p} \\ \tilde{u} \\ \tilde{\alpha} \end{array} \right\} = \left[\begin{array}{lll} 0 & e^{i\kappa_2 s} & e^{i\kappa_3 s} \\ 0 & -e^{i\kappa_2 s}/\bar{\rho}c & e^{i\kappa_3 s}/\bar{\rho}c \\ e^{i\kappa_1 s} & -(1-\bar{\alpha})e^{i\kappa_2 s}/\bar{\rho}c^2 & -(1-\bar{\alpha})e^{i\kappa_3 s}/\bar{\rho}c^2 \end{array} \right] \left\{ \begin{array}{c} A_1 \\ P_2 \\ P_3 \end{array} \right\} \qquad \text{(Nrm12)}$$

and the transfer function between two locations $s = s_1$ and $s = s_2$ follows by eliminating the vector $\{A_1, P_2, P_3\}$ from the expressions (Nrm12) for the state vectors at those two locations.

Transfer function methods for multiphase flow are nowhere near as well developed as they are for single phase flows but, given the number and ubiquity of instability problems in multiphase flows (Ishii 1982), it is inevitable that these methods will gradually develop into a tool that is useful in a wide spectrum of applications.