Nozzle Flows

The barotropic relations of the last section can be used in conjunction with the steady, one-dimensional continuity and frictionless momentum equations,

$$\frac{d}{ds}(\rho A u) = 0 \tag{Nlf1}$$

and

$$u\frac{du}{ds} = -\frac{1}{\rho}\frac{dp}{ds} \tag{Nlf2}$$

to synthesize homogeneous multiphase flow in ducts and nozzles. The predicted phenomena are qualitatively similar to those in one-dimensional gas dynamics. The results for isothermal, two-component flow were first detailed by Tangren, Dodge, and Seifert (1949); more general results for any polytropic index are given in this section.

Using the barotropic relation given by equation (Nle2) and equation (Nle1) for the mixture density, ρ, to eliminate p and ρ from the momentum equation (Nlf2), one obtains

$$u\ du = \frac{kp_o}{\rho_o}\ \frac{\alpha_o^k}{(1-\alpha_o)^{k-1}}\ \frac{(1-\alpha)^{k-2}}{\alpha^{k+1}}d\alpha \tag{Nlf3}$$

which upon integration and imposition of the reservoir condition, $u_o = 0$, yields

$$u^2 = \frac{2kp_o}{\rho_o}\ \frac{\alpha_o^k}{(1-\alpha_o)^{k-1}}\left[\frac{1}{k}\left\{\left(\frac{1-\alpha_o}{\alpha_o}\right)^k - \left(\frac{1-\alpha}{\alpha}\right)^k\right\} + \right.$$

$$\text{either} \qquad \frac{1}{(k-1)}\left\{\left(\frac{1-\alpha_o}{\alpha_o}\right)^{k-1} - \left(\frac{1-\alpha}{\alpha}\right)^{k-1}\right\}\Bigg] \quad \text{if } k \neq 1$$

$$\text{or} \qquad ln\left\{\frac{(1-\alpha_o)\alpha}{\alpha_o(1-\alpha)}\right\}\Bigg] \quad \text{if } k = 1 \tag{Nlf4}$$

Given the reservoir conditions p_o and α_o as well as the polytropic index k and the liquid density (assumed constant), this relates the velocity, u, at any position in the duct to the gas volume fraction, α, at that location. The pressure, p, density, ρ, and volume fraction, α, are related by equations (Nle1) and (Nle2). The continuity equation,

$$A = \text{Constant}/\rho u = \text{Constant}/u(1-\alpha) \tag{Nlf5}$$

completes the system of equations by permitting identification of the location where p, ρ, u, and α occur from knowledge of the cross-sectional area, A.

As in gas dynamics the conditions at a throat play a particular role in determining both the overall flow and the mass flow rate. This results from the observation that equations (Nlf1) and (Nle6) may be combined to obtain

$$\frac{1}{A}\ \frac{dA}{ds} = \frac{1}{\rho}\ \frac{dp}{ds}\left(\frac{1}{u^2} - \frac{1}{c^2}\right) \tag{Nlf6}$$

where $c^2 = dp/d\rho$. Hence at a throat where $dA/ds = 0$: either $dp/ds = 0$, which is true when the flow is entirely subsonic and unchoked; or $u = c$, which is true when the flow is choked. Denoting choked

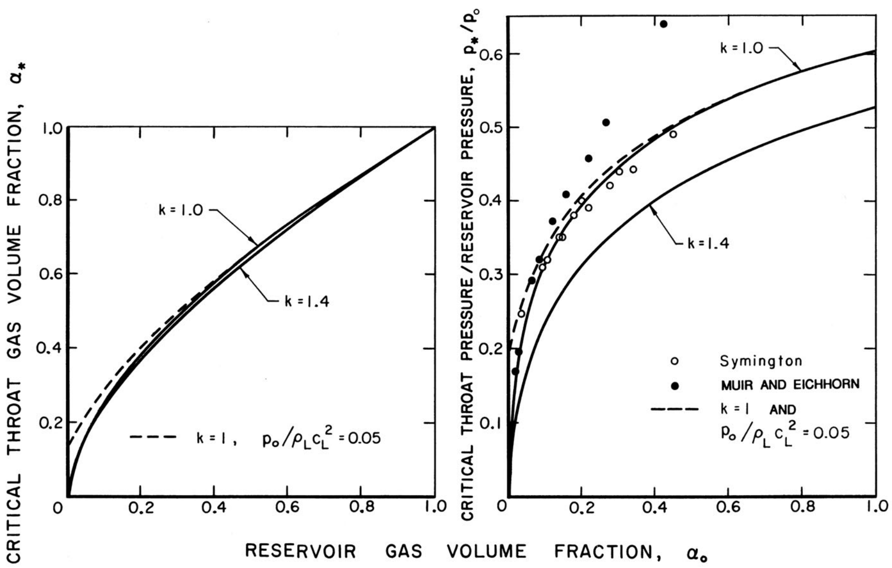

Figure 1: Critical or choked flow throat characteristics for the flow of a two-component gas/liquid mixture through a nozzle. On the left is the throat gas volume fraction as a function of the reservoir gas volume fraction, α_o, for gas polytropic indices of $k = 1.0$ and 1.4 and an incompressible liquid (solid lines) and for $k = 1$ and a compressible liquid with $p_o/\rho_L c_L^2 = 0.05$ (dashed line). On the right are the corresponding ratios of critical throat pressure to reservoir pressure. Also shown is the experimental data of Symington (1978) and Muir and Eichhorn (1963).

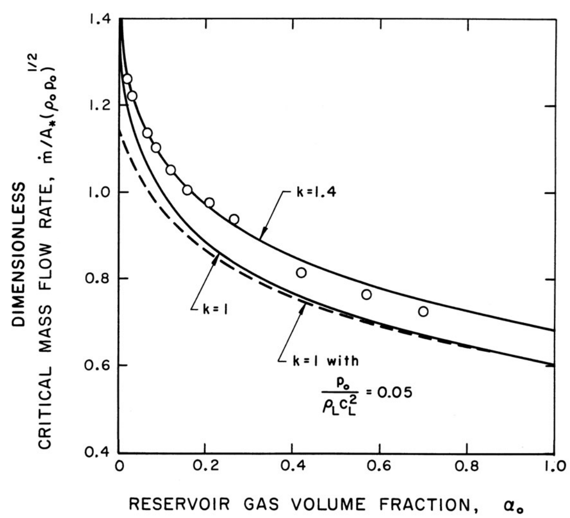

Figure 2: Dimensionless critical mass flow rate, $\dot{m}/A_*(p_o\rho_o)^{\frac{1}{2}}$, as a function of α_o for choked flow of a gas/liquid flow through a nozzle. Solid lines are incompressible liquid results for polytropic indices of 1.4 and 1.0. Dashed line shows effect of liquid compressibility for $p_o/\rho_L c_L^2 = 0.05$. The experimental data ($\odot$) are from Muir and Eichhorn (1963).

conditions at a throat by the subscript $*$, it follows by equating the right-hand sides of equations (Nle3) and (Nlf4) that the gas volume fraction at the throat, α_*, must be given when $k \neq 1$ by the solution of

$$\frac{(1-\alpha_*)^{k-1}}{2\alpha_*^{k+1}} = \frac{1}{k}\left\{\left(\frac{1-\alpha_o}{\alpha_o}\right)^k - \left(\frac{1-\alpha_*}{\alpha_*}\right)^k\right\}$$

$$+\frac{1}{(k-1)}\left\{\left(\frac{1-\alpha_o}{\alpha_o}\right)^{k-1} - \left(\frac{1-\alpha_*}{\alpha_*}\right)^{k-1}\right\} \tag{Nlf7}$$

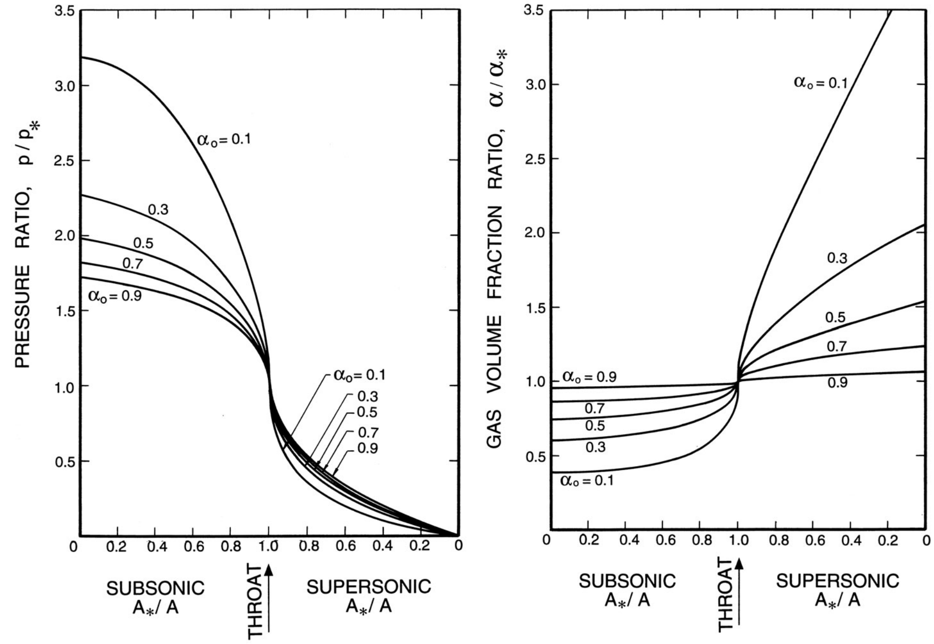

Figure 3: Left: Ratio of the pressure, p, to the throat pressure, p_*, and Right: Ratio of the void fraction, α, to the throat void fraction, α_*, for two-component flow in a duct with isothermal gas behavior.

or, in the case of isothermal gas behavior ($k = 1$), by the solution of

$$\frac{1}{2\alpha_*^2} = \frac{1}{\alpha_o} - \frac{1}{\alpha_*} + ln\left\{\frac{(1-\alpha_o)\alpha_*}{\alpha_o(1-\alpha_*)}\right\} \qquad \text{(Nlf8)}$$

Thus the throat gas volume fraction, α_*, under choked flow conditions is a function only of the reservoir gas volume fraction, α_o, and the polytropic index. Solutions of equations (Nlf7) and (Nlf8) for two typical cases, $k = 1.4$ and $k = 1.0$, are shown in figure 1. The corresponding ratio of the choked throat pressure, p_*, to the reservoir pressure, p_o, follows immediately from equation (Nle2) given $\alpha = \alpha_*$ and is also shown in figure 1. Finally, the choked mass flow rate, $\dot{m}$, follows as $\rho_* A_* c_*$ where A_* is the cross-sectional area of the throat and

$$\frac{\dot{m}}{A_*(p_o\rho_o)^{\frac{1}{2}}} = k^{\frac{1}{2}} \frac{\alpha_o^{\frac{k}{2}}}{(1-\alpha_o)^{\frac{k+1}{2}}} \left(\frac{1-\alpha_*}{\alpha_*}\right)^{\frac{k+1}{2}} \qquad \text{(Nlf9)}$$

This dimensionless choked mass flow rate is exhibited in figure 2 for $k = 1.4$ and $k = 1$.

Data from the experiments of Symington (1978) and Muir and Eichhorn (1963) are included in figures 1 and 2. Symington's data on the critical pressure ratio (figure 1) is in good agreement with the isothermal ($k = 1$) analysis indicating that, at least in his experiments, the heat transfer between the bubbles and the liquid is large enough to maintain constant gas temperature in the bubbles. On the other hand, the experiments of Muir and Eichhorn yielded larger critical pressure ratios and flow rates than the isothermal theory. However, Muir and Eichhorn measured significant slip between the bubbles and the liquid (strictly speaking the abscissa for their data in figures 1 and 2 should be the upstream volumetric quality rather than the void fraction), and the discrepancy could be due to the errors introduced into the present analysis by the neglect of possible relative motion (see also van Wijngaarden 1972).

Finally, the pressure, volume fraction, and velocity elsewhere in the duct or nozzle can be related to the throat conditions and the ratio of the area, A, to the throat area, A_*. These relations, which are presented

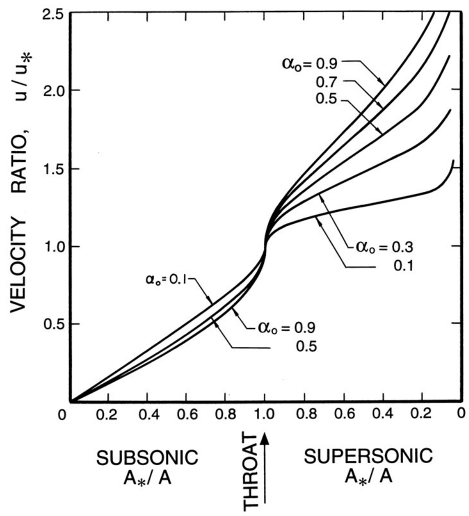

Figure 4: Ratio of the velocity, u, to the throat velocity, u_*, for two-component flow in a duct with isothermal gas behavior.

in figures 3 and 4 for the case $k = 1$ and various reservoir volume fractions, α_o, are most readily obtained in the following manner. Given α_o and k, p_*/p_o and α_* follow from figure 1. Then for p/p_o or p/p_*, α and u follow from equations (Nle2) and (Nlf4) and the corresponding A/A_* follows by using equation (Nlf5). The resulting charts, figures 3 and 4, can then be used in the same way as the corresponding graphs in gas dynamics.

If the gas volume fraction, α_o, is sufficiently small so that it is comparable with $p_o/\rho_L c_L^2$, then the barotropic equation (Nle4) should be used instead of equation (Nle2). In cases like this in which it is sufficient to assume that $\rho \approx \rho_L(1-\alpha)$, integration of the momentum equation (Nlf2) is most readily accomplished by writing it in the form

$$\frac{\rho_L}{p_o}\frac{u^2}{2} = 1 - \frac{p}{p_o} + \int_{p/p_o}^{1} \left(\frac{\alpha}{1-\alpha}\right) d\left(\frac{p}{p_o}\right) \tag{Nlf10}$$

Then substitution of equation (Nle4) for $\alpha/(1-\alpha)$ leads in the present case to

$$u^2 = \frac{2p_o}{\rho_L}\left[1 - \frac{p}{p_o} + \frac{k}{2(k+1)}\,\frac{p_o}{\rho_L c_L^2}\left\{\frac{p^2}{p_o^2} - 1\right\} + \right.$$

$$\text{either} \quad \left.\frac{k}{(k-1)}\left\{\frac{\alpha_o}{1-\alpha_o} + \frac{k}{(k+1)}\,\frac{p_o}{\rho_L c_L^2}\right\}\left\{1 - \left(\frac{p}{p_o}\right)^{\frac{k-1}{k}}\right\}\right] \quad \text{for } k \neq 1$$

$$\text{or} \quad \left.\left\{\frac{\alpha_o}{1-\alpha_o} + \frac{1}{2}\frac{p_o}{\rho_L c_L^2}\right\} ln\left(\frac{p_o}{p}\right)\right] \quad \text{for } k = 1 \tag{Nlf11}$$

The throat pressure, p_* (or rather p_*/p_o), is then obtained by equating the velocity u for $p = p_*$ from equation (Nlf11) to the sonic velocity c at $p = p_*$ obtained from equation (Nle5). The resulting relation, though algebraically complicated, is readily solved for the critical pressure ratio, p_*/p_o, and the throat gas volume fraction, α_*, follows from equation (Nle4). Values of p_*/p_o for $k = 1$ and $k = 1.4$ are shown in figure 1 for the particular value of $p_o/\rho_L c_L^2$ of 0.05. Note that the most significant deviations caused by liquid compressibility occur for gas volume fractions of the order of 0.05 or less. The corresponding dimensionless critical mass flow rates, $\dot{m}/A_*(\rho_o p_o)^{\frac{1}{2}}$, are also readily calculated from

$$\frac{\dot{m}}{A_*(\rho_o p_o)^{\frac{1}{2}}} = \frac{(1-\alpha_*)c_*}{[p_o(1-\alpha_o)/\rho_L]^{\frac{1}{2}}} \tag{Nlf12}$$

and sample results are shown in figure 2.