Barotropic Relations

Conceptually, the expressions for the sonic velocity, equations (Nlb9), (Nlb10), (Nlb11), or (Nld9), need only be integrated (after substituting $c^2 = dp/d\rho$) in order to obtain the barotropic relation, $p(\rho)$, for the mixture. In practice this is algebraically complicated except for some of the simpler forms for c^2.

Consider first the case of the two-component mixture in the absence of mass exchange or surface tension as given by equation (Nlb10). It will initially be assumed that the gas volume fraction is not too small so that equation (Nlb11) can be used; we will return later to the case of small gas volume fraction. It is also assumed that the liquid or solid density, ρ_L, is constant and that $p \propto \rho_G^k$. Furthermore it is convenient, as in gas dynamics, to choose reservoir conditions, $p = p_o, \alpha = \alpha_o, \rho_G = \rho_{Go}$ to establish the integration constants. Then it follows from the integration of equation (Nlb11) that

$$\rho = \rho_o(1-\alpha)\big/(1-\alpha_o) \qquad \text{(Nle1)}$$

and that

$$\frac{p}{p_o} = \left[\frac{\alpha_o(1-\alpha)}{(1-\alpha_o)\alpha}\right]^k = \left[\frac{\alpha_o\rho}{\rho_o-(1-\alpha_o)\rho}\right]^k \qquad \text{(Nle2)}$$

where $\rho_o = \rho_L(1-\alpha_o) + \rho_{Go}\alpha_o$. It also follows that, written in terms of α,

$$c^2 = \frac{kp_o}{\rho_o}\;\frac{(1-\alpha)^{k-1}}{\alpha^{k+1}}\;\frac{\alpha_o^k}{(1-\alpha_o)^{k-1}} \qquad \text{(Nle3)}$$

As will be discussed later, Tangren, Dodge, and Seifert (1949) first made use of a more limited form of the barotropic relation of equation (Nle2) to evaluate the one-dimensional flow of gas/liquid mixtures in ducts and nozzles.

In the case of very small gas volume fractions, α, it may be necessary to include the liquid compressibility term, $1-\alpha/\rho_L c_L^2$, in equation (Nlb10). Exact integration then becomes very complicated. However, it is sufficiently accurate at small gas volume fractions to approximate the mixture density ρ by $\rho_L(1-\alpha)$, and then integration (assuming $\rho_L c_L^2$ = constant) yields

$$\frac{\alpha}{(1-\alpha)} = \left[\frac{\alpha_o}{(1-\alpha_o)} + \frac{k}{(k+1)}\;\frac{p_o}{\rho_L c_L^2}\right]\left(\frac{p_o}{p}\right)^{\frac{1}{k}} - \frac{k}{(k+1)}\;\frac{p_o}{\rho_L c_L^2}\;\frac{p}{p_o} \qquad \text{(Nle4)}$$

and the sonic velocity can be expressed in terms of p/p_o alone by using equation (Nle4) and noting that

$$c^2 = \frac{p}{\rho_L}\frac{\left[1+\frac{\alpha}{(1-\alpha)}\right]^2}{\left[\frac{1}{k}\frac{\alpha}{(1-\alpha)} + \frac{p}{\rho_L c_L^2}\right]} \qquad \text{(Nle5)}$$

Implicit within equation (Nle4) is the barotropic relation, $p(\alpha)$, analogous to equation (Nle2). Note that equation (Nle4) reduces to equation (Nle2) when $p_o/\rho_L c_L^2$ is set equal to zero. Indeed, it is clear from equation (Nle4) that the liquid compressibility has a negligible effect only if $\alpha_o \gg p_o/\rho_L c_L^2$. This parameter, $p_o/\rho_L c_L^2$, is usually quite small. For example, for saturated water at $5\times10^7\ kg/msec^2$ (500 psi) the value of $p_o/\rho_L c_L^2$ is approximately 0.03. Nevertheless, there are many practical problems in which one is concerned with the discharge of a predominantly liquid medium from high pressure containers, and under these circumstances it can be important to include the liquid compressibility effects.

Now turning attention to a two-phase rather than two-component homogeneous mixture, the particular form of the sonic velocity given in equation (Nld10) may be integrated to yield the implicit barotropic relation

$$\frac{\alpha}{1-\alpha} = \left[\frac{\alpha_o}{(1-\alpha_o)} + \frac{k_L p_o^{-\eta}}{(k_V-\eta)}\right]\left(\frac{p_o}{p}\right)^{k_V} - \left[\frac{k_L p_o^{-\eta}}{(k_V-\eta)}\right]\left(\frac{p_o}{p}\right)^{\eta} \tag{Nle6}$$

in which the approximation $\rho \approx \rho_L(1-\alpha)$ has been used. As before, c^2 may be expressed in terms of p/p_o alone by noting that

$$c^2 = \frac{p}{\rho_L}\frac{\left[1+\frac{\alpha}{1-\alpha}\right]^2}{\left[k_V\frac{\alpha}{(1-\alpha)} + k_L p^{-\eta}\right]} \tag{Nle7}$$

Finally, we note that close to $\alpha = 1$ the equations (Nle6) and (Nle7) may fail because the approximation $\rho \approx \rho_L(1-\alpha)$ is not sufficiently accurate.