Steady Planar Flows

The classic free streamline solution for an arbitrary finite body with a fully developed cavity is obtained by mapping both the geometry of the physical plane (z-plane, section (Nub)) and the geometry of the f-plane (Figure 1) into the lower half of a parametric, ζ-plane. The wetted surface is mapped onto the interval, $\eta = 0$, $-1 < \xi < 1$ and the stagnation point, 0, is mapped into the origin. For the first three closure models of section (Nub), the geometries of the corresponding ζ-planes are sketched in Figure 1. The $f = f(\zeta)$ mapping follows from the generalized Schwarz-Christoffel transformation (Gilbarg 1949); for the three closure models this yields respectively

$$\frac{df}{d\zeta} = \frac{C\zeta}{(\zeta-\zeta_I)^{\frac{3}{2}}(\zeta-\bar{\zeta}_I)^{\frac{3}{2}}} \tag{Nue1}$$

$$\frac{df}{d\zeta} = \frac{C\zeta(\zeta-\zeta_C)}{(\zeta-\zeta_I)^2(\zeta-\bar{\zeta}_I)^2} \tag{Nue2}$$

$$\frac{df}{d\zeta} = \frac{C\zeta(\zeta-\zeta_R)(\zeta-\bar{\zeta}_R)}{(\zeta-\zeta_I)^2(\zeta-\bar{\zeta}_I)^2(\zeta-\zeta_J)} \tag{Nue3}$$

where C is a real constant, ζ_I is the value of ζ at the point I (the point at infinity in the z-plane), ζ_C is the value of ζ at the end of the constant velocity part of the free streamlines, and ζ_R and ζ_J are the values at the rear stagnation point and the upstream infinity point in the reentrant jet model.

The wetted surface, AOB, will be given parametrically by $x(s), y(s)$ where s is the distance measured along that surface from the point A. Then the boundary conditions on the logarithmic hodograph variable, $\varpi = \chi + i\theta$, are

$$\theta^-(\xi) \equiv \theta(\xi,0^-) = \pi F(-\xi) + \theta^*(s(\xi)) \quad \text{on} \quad -1 < \xi < 1 \tag{Nue4}$$

$$\chi^-(\xi) \equiv \chi(\xi,0^-) = 0 \quad \text{on} \quad \xi > 1 \quad \text{and} \quad \xi < -1 \tag{Nue5}$$

where the superscripts $+$ and $-$ will be used to denote values on the ξ axis of the ζ-plane just above and just below the cut. The function $F(-\xi)$ takes a value of 1 for $\xi < 0$ and a value of 0 for $\xi > 0$. The function $\theta^*(s)$ is the inclination of the wetted surface so that $\tan\theta^* = dy/ds\big/dx/ds$. The solution to the above Reimann-Hilbert problem is

$$\varpi(\zeta) = \varpi_0(\zeta) + \varpi_1(\zeta) + \varpi_2(\zeta) \tag{Nue6}$$

where

$$\varpi_0(\zeta) = \log\left\{\left(1 + i(\zeta^2-1)^{\frac{1}{2}}\right)/\zeta\right\} \tag{Nue7}$$

$$\varpi_1(\zeta) = \frac{1}{i\pi}\int_{-1}^{1}\left(\frac{\zeta^2-1}{1-\beta^2}\right)^{\frac{1}{2}}\frac{\theta^{**}(\beta)d\beta}{\beta-\zeta} \tag{Nue8}$$

where β is a dummy variable, $\theta^{**}(\xi) = \theta^*(s(\xi))$ and the function $(\zeta^2-1)^{\frac{1}{2}}$ is analytic in the ζ-plane cut along the ξ axis from -1 to $+1$ so that it tends to ζ as $|\zeta| \to \infty$. The third function, $\varpi_2(\zeta)$, is zero for the Riabouchinsky and open-wake closure models; it is only required for the reentrant jet model and, in that case,

$$\varpi_2(\zeta) = \log\left\{\frac{(\beta-\bar{\beta}_R)(\beta\beta_R-1)}{(\beta-\beta_R)(\beta\bar{\beta}_R-1)}\right\} \quad \text{where} \quad \zeta = (\beta+\beta^{-1})/2 \tag{Nue9}$$

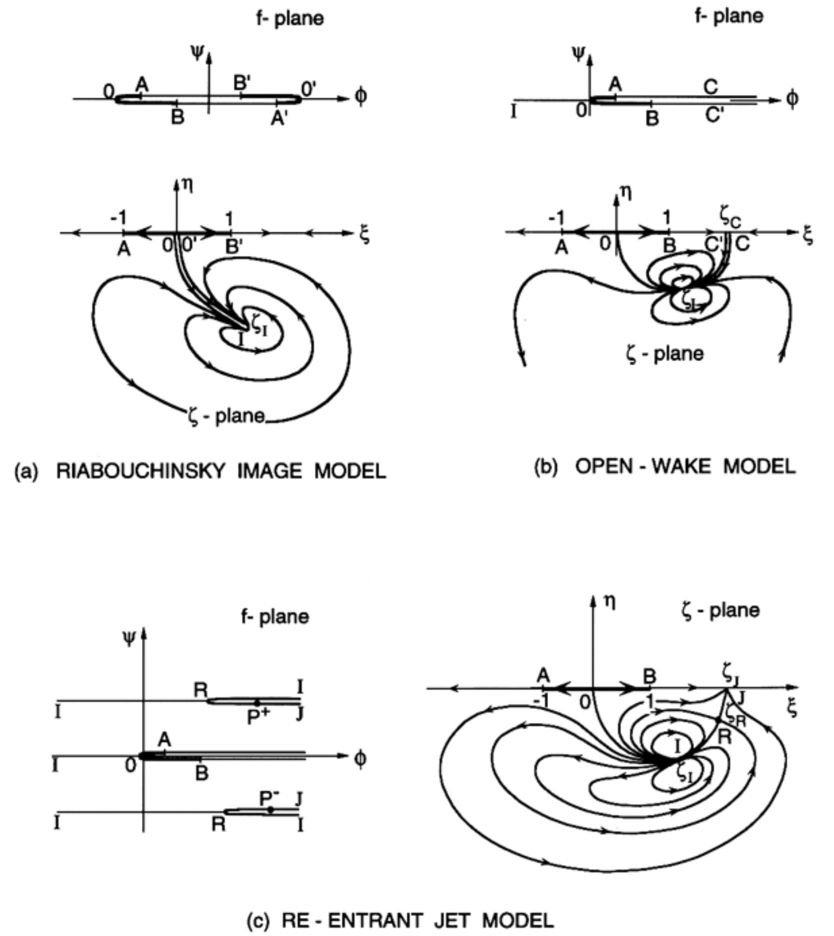

Figure 1: Streamlines in the complex potential f-plane and the parametric ζ-plane where the flow boundaries and points correspond to those of section (Nub).

Given $\varpi(\zeta)$, the physical coordinate $z(\zeta)$ is then calculated using

$$z(\zeta) = \int \frac{1}{w(\zeta)} \frac{df}{d\zeta} d\zeta \tag{Nue10}$$

The distance along the wetted surface from the point A is given by

$$s(\xi) = \frac{C}{q_c} \int_{-1}^{\xi} e^{\Gamma_1(\xi)} \Gamma_2(\xi) d\xi \tag{Nue11}$$

where

$$\Gamma_1(\xi) = -\frac{1}{\pi} \oint_{-1}^{1} \left(\frac{1-\xi^2}{1-\beta^2} \right)^{\frac{1}{2}} \frac{\theta^{**}(\beta) d\beta}{\beta - \xi} \tag{Nue12}$$

$$\Gamma_2(\xi) = \frac{1}{C} \exp\left\{ \varpi_0^-(\xi) + \varpi_2^-(\xi) \right\} \frac{df}{d\xi} \tag{Nue13}$$

where the integral in equation (Nue12) takes its Cauchy principal value.

Now consider the conditions that can be applied to evaluate the unknown parameters in the problem, namely C and ζ_I in the case of the Riabouchinsky model, C, ζ_I, and ζ_C in the case of the open-wake model, and C, ζ_I, ζ_R, and ζ_J in the case of the reentrant jet model. All three models require that the total wetted surface length, $s(1)$, be equal to a known value, and this establishes the length scale in the flow. They also require that the velocity at $z \rightarrow \infty$ have the known magnitude, U_∞, and a given inclination, α, to the chord, AB. Consequently this condition becomes

$$\varpi(\zeta_0) = \frac{1}{2}\log(1+\sigma) + i\alpha \tag{Nue14}$$

This is sufficient to determine the solution for the Riabouchinsky model. Additional conditions for the open-wake model can be derived from the fact that $f(\zeta)$ must be simply covered in the vicinity of ζ_0 and, for the reentrant jet model, that $z(\zeta)$ must be simply covered in the vicinity of ζ_0. Also the circulation around the cavity can be freely chosen in the re-entrant jet model. Finally, if the free streamline detachment is smooth and therefore initially unknown, its location must be established using the Brillouin-Villat condition (see section (Nub)).

As is the case with all steady planar potential flows involving a body in an infinite uniform stream, the behavior of the complex velocity, $w(z)$, far from the body can be particularly revealing. If $w(z)$ is expanded in powers of $1/z$ then

$$w(z) = U_\infty e^{-i\alpha} + \frac{Q+i\Gamma}{2\pi}\frac{1}{z} + (C_1 + iC_2)\frac{1}{z^2} + O(\frac{1}{z^3}) \tag{Nue15}$$

where U_∞ and α are the magnitude and inclination of the free stream. The quantity Q is the net source strength required to simulate the body-cavity system and must therefore be zero for a finite body-cavity. This constitutes a cavity closure condition. The quantity, Γ, is the circulation around the body-cavity so that the lift is given by $\rho U_\infty \Gamma$. Evaluation of the $1/z$ term far from the body provides the simplest way to evaluate the lift.

The mathematical detail involved in producing results from these solutions (Wu and Wang 1964b) is considerable except for simple symmetric bodies. For more complex, bluff bodies it is probably more efficient to resort to one of the modern numerical methods (for example a panel method) rather than to attempt to sort through all the complex algebra of the above solutions. For streamlined bodies, a third alternative is the algebraically simpler linear theory for cavity flow, which is briefly reviewed in section (Nug). There are, however, a number of valuable results that can be obtained from the above exact, nonlinear theory, and we will examine just a few of these in the next section.