Small Slip Perturbation Analysis

The analyses described in the preceding sections, (Nnf), (Nng) and (Nnh), used a linearization about a uniform and constant mean state and assumed that the perturbations in the variables were small compared with their mean values. Another, different linearization known as the small slip approximation can be advantageous in other contexts in which the mean state is more complicated. It proceeds as follows. First recall that the solutions always asymptote to those for a single effective gas when t_u and t_T tend to zero. Therefore, when these quantities are small and the slip between the particles and the gas is correspondingly small, we can consider constructing solutions in which the flow variables are represented by power series expansions in one of these small quantities, say t_u, and it is assumed that the other (t_T) is of similar order. Then, generically,

$$Q(x_i,t) = Q^{(0)}(x_i,t) + t_u Q^{(1)}(x_i,t) + t_u^2 Q^{(2)}(x_i,t) + ... \qquad \text{(Nni1)}$$

where Q represents any of the flow quantities, u_{Ci}, u_{Di}, T_C, T_D, p, ρ_C, α_C, α_D, etc. In addition, it is assumed for the reasons given above that the slip velocity and slip temperature, $(u_{Ci} - u_{Di})$ and $(T_C - T_D)$, are of order t_u so that

$$u_{Ci}^{(0)} = u_{Di}^{(0)} = u_i^{(0)} \quad ; \quad T_C^{(0)} = T_D^{(0)} = T^{(0)} \qquad \text{(Nni2)}$$

Substituting these expansions into the basic equations (Nnb6), (Nnb7) and (Nnb8) and gathering together the terms of like order in t_u we obtain the following zeroth order continuity, momentum and energy relations (omitting gravity):

$$\frac{\partial}{\partial x_i}\left((1+\xi)\rho_C^{(0)} u_i^{(0)}\right) = 0 \qquad \text{(Nni3)}$$

$$(1+\xi)\rho_C^{(0)} u_k^{(0)} \frac{\partial u_i^{(0)}}{\partial x_k} = -\frac{\partial p^{(0)}}{\partial x_i} + \frac{\partial \sigma_{Cik}^{D(0)}}{\partial x_k} \qquad \text{(Nni4)}$$

$$\rho_C^{(0)} u_k^{(0)} (c_{pC} + \xi c_{sD}) \frac{\partial T^{(0)}}{\partial x_k} = u_k^{(0)} \frac{\partial p^{(0)}}{\partial x_k} + \sigma_{Cik}^{D(0)} \frac{\partial u_i^{(0)}}{\partial x_k} \qquad \text{(Nni5)}$$

Note that Marble (1970) also includes thermal conduction in the energy equation. Clearly the above are just the equations for single phase flow of the effective gas defined in section (Nnc). Conventional single phase gas dynamic methods can therefore be deployed to obtain their solution.

Next, the relaxation equations (Nnd3) and (Nnd6) that are first order in t_u yield:

$$u_k^{(0)} \frac{\partial u_i^{(0)}}{\partial x_k} = u_{Ci}^{(1)} - u_{Di}^{(1)} \qquad \text{(Nni6)}$$

$$u_k^{(0)} \frac{\partial T^{(0)}}{\partial x_k} = \left(\frac{t_u}{t_T}\right)\left(\frac{Nu}{2}\right)(T_C^{(1)} - T_D^{(1)}) \qquad \text{(Nni7)}$$

From these the slip velocity and slip temperature can be calculated once the zeroth order solution is known.

The third step is to evaluate the modification to the effective gas solution caused by the slip velocity and temperature; in other words, to evaluate the first order terms, $u_{Ci}^{(1)}$, $T_C^{(1)}$, etc. The relations for these are derived by extracting the $O(t_u)$ terms from the continuity, momentum and energy equations. For example, the continuity equation yields

$$\frac{\partial}{\partial x_i}\left[\xi\rho_C^{(0)}(u_{Ci}^{(1)} - u_{Di}^{(1)}) + u_i^{(0)}(\rho_D\alpha_D^{(1)} - \xi\rho_C^{(1)})\right] = 0 \qquad \text{(Nni8)}$$

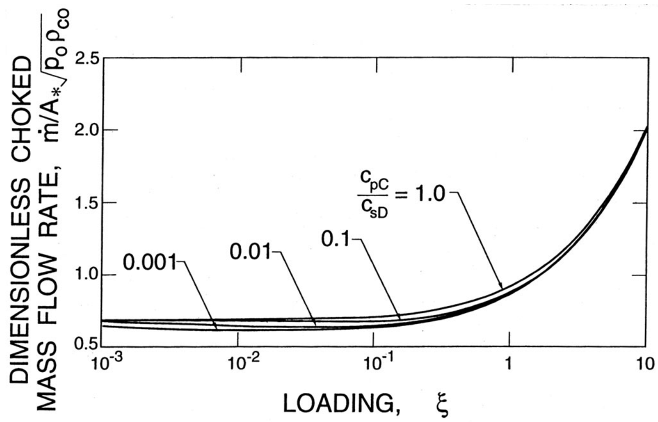

Figure 1: The dimensionless choked mass flow rate as a function of loading, ξ, for $\gamma_C = 1.4$ and various specific heat ratios, c_{pC}/c_{sD} as shown.

This and the corresponding first order momentum and energy equations can then be solved to find the $O(t_u)$ slip perturbations to the gas and particle flow variables. For further details the reader is referred to Marble (1970).

A particular useful application of the slip perturbation method is to the one-dimensional steady flow in a convergent/divergent nozzle. The zeroth order, effective gas solution leads to pressure, velocity, temperature and density profiles that are straightforward functions of the Mach number which is, in turn, derived from the cross-sectional area. This area is used as a surrogate axial coordinate. Here we focus on just one part of this solution namely the choked mass flow rate, $\dot{m}$, that, according to the single phase, effective gas analysis will be given by

$$\frac{\dot{m}}{A_*(p_0\rho_{C0})^{\frac{1}{2}}} = (1+\xi)^{\frac{1}{2}}\gamma^{\frac{1}{2}}\left(\frac{2}{1+\gamma}\right)^{(\gamma+1)/2(\gamma-1)} \qquad \text{(Nni9)}$$

where p_0 and ρ_{C0} refer to the pressure and gas density in the upstream reservoir, A_* is the throat cross-sectional area and γ is the effective specific heat ratio as given in equation (Nnc6). The dimensionless choked mass flow rate on the left of equation (Nni9) is a function only of ξ, γ_C and the specific heat ratio, c_{pC}/c_{sD}. As shown in figure 1, this is primarily a function of the loading ξ and is only weakly dependent on the specific heat ratio.