Natural Modes of a Spherical Bubble Cloud

A second illustrative example of the effect of bubble dynamics on the behavior of a homogeneous bubbly mixture is the study of the dynamics of a finite cloud of bubbles. One of the earliest investigations of the collective dynamics of bubble clouds was the work of van Wijngaarden (1964) on the oscillations of a layer of bubbles near a wall. Later d'Agostino and Brennen (1983) investigated the dynamics of a spherical cloud (see also d'Agostino and Brennen 1989, Omta 1987), and we will choose the latter as a example of that class of problems with one space dimension in which analytical solutions may be obtained but only after linearization of the Rayleigh-Plesset equation (Nmb3).

The geometry of the spherical cloud is shown in figure 1. Within the cloud of radius, $A(t)$, the population of bubbles per unit *liquid* volume, η, is assumed constant and uniform. The linearization assumes small perturbations of the bubbles from an equilibrium radius, R_o:

$$R(r,t) = R_o\left[1+\varphi(r,t)\right] \ , \ |\varphi| \ll 1 \tag{Nmh1}$$

We will seek the response of the cloud to a correspondingly small perturbation in the pressure at infinity, $p_\infty(t)$, that is represented by

$$p_\infty(t) = p(\infty,t) = \bar{p} + Re\left\{\tilde{p}e^{i\omega t}\right\} \tag{Nmh2}$$

where $\bar{p}$ is the mean, uniform pressure and $\tilde{p}$ and ω are the perturbation amplitude and frequency, respectively. The solution will relate the pressure, $p(r,t)$, radial velocity, $u(r,t)$, void fraction, $\alpha(r,t)$, and bubble perturbation, $\varphi(r,t)$, to $\tilde{p}$. Since the analysis is linear, the response to excitation involving multiple frequencies can be obtained by Fourier synthesis.

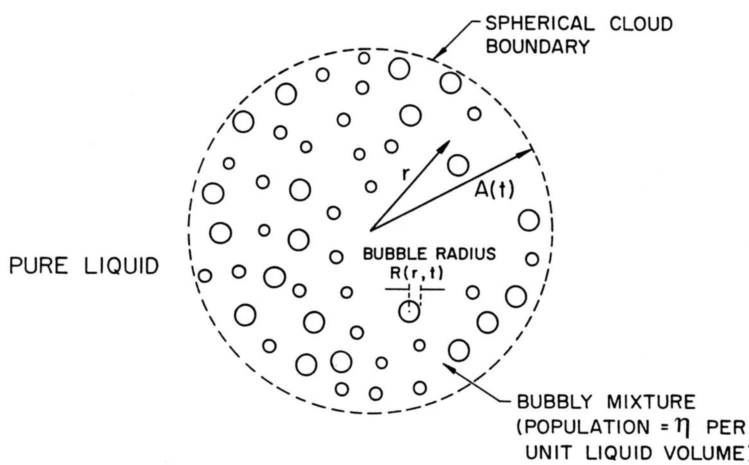

Figure 1: Notation for the analysis of a spherical cloud of bubbles.

One further restriction is necessary in order to linearize the governing equations (Nmb1), (Nmb2) and (Nmb3). It is assumed that the mean void fraction in the cloud, α_o, is small so that the term $(1+\eta v)$ in equations (Nmb1) and (Nmb2) is approximately unity. Then these equations become

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) = \eta\frac{Dv}{Dt} \tag{Nmh3}$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{Nmh4}$$

It is readily shown that the velocity u is of order φ and hence the convective component of the material derivative is of order φ^2; thus the linearization implies replacing D/Dt by $\partial/\partial t$. Then to order φ the Rayleigh-Plesset equation yields

$$p(r,t) = \bar{p} - \rho R_o^2 \left[\frac{\partial^2 \varphi}{\partial t^2} + \omega_n^2 \varphi\right] \quad ; \quad r < A(t) \tag{Nmh5}$$

where ω_n is the natural frequency of an individual bubble if it were alone in an infinite fluid (equation (Nmc10)). It must be assumed that the bubbles are in stable equilibrium in the mean state so that ω_n is real.

Upon substitution of equations (Nmh1) and (Nmh5) into (Nmh3) and (Nmh4) and elimination of $u(r,t)$ one obtains the following equation for $\varphi(r,t)$ in the domain $r < A(t)$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\left\{\frac{\partial^2 \varphi}{\partial t^2} + \omega_n^2 \varphi\right\}\right] - 4\pi\eta R_o \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{Nmh6}$$

The incompressible liquid flow outside the cloud, $r \geq A(t)$, must have the standard solution of the form:

$$u(r,t) = \frac{C(t)}{r^2} \; ; \; r \geq A(t) \tag{Nmh7}$$

$$p(r,t) = p_\infty(t) + \frac{\rho}{r}\frac{dC(t)}{dt} - \frac{\rho C^2}{2r^4} \; ; \; r \geq A(t) \tag{Nmh8}$$

where $C(t)$ is of perturbation order. It follows that, to the first order in $\varphi(r,t)$, the continuity of $u(r,t)$ and $p(r,t)$ at the interface between the cloud and the pure liquid leads to the following boundary condition for $\varphi(r,t)$:

$$\left(1 + A_o \frac{\partial}{\partial r}\right)\left[\frac{\partial^2 \varphi}{\partial t^2} + \omega_n^2 \varphi\right]_{r=A_o} = \frac{\bar{p} - p_\infty(t)}{\rho R_o^2} \tag{Nmh9}$$

The solution of equation (Nmh6) under the above boundary condition is

$$\varphi(r,t) = -\frac{1}{\rho R_o^2} Re\left\{\frac{\tilde{p}}{\omega_n^2 - \omega^2}\;\frac{e^{i\omega t}}{\cos \lambda A_o}\;\frac{\sin \lambda r}{\lambda r}\right\} \quad ; \quad r < A_o \tag{Nmh10}$$

where:

$$\lambda^2 = 4\pi\eta R_o \frac{\omega^2}{\omega_n^2 - \omega^2} \tag{Nmh11}$$

Another possible solution involving $(\cos \lambda r)/\lambda r$ has been eliminated since $\varphi(r,t)$ must clearly be finite as $r \to 0$. Therefore in the domain $r < A_o$:

$$R(r,t) = R_o - \frac{1}{\rho R_o} Re\left\{\frac{\tilde{p}}{\omega_n^2 - \omega^2}\;\frac{e^{i\omega t}}{\cos \lambda A_o}\;\frac{\sin \lambda r}{\lambda r}\right\} \tag{Nmh12}$$

$$u(r,t) = \frac{1}{\rho} Re\left\{i\frac{\tilde{p}}{\omega}\;\frac{1}{r}\left(\frac{\sin \lambda r}{\lambda r} - \cos \lambda r\right)\frac{e^{i\omega t}}{\cos \lambda A_o}\right\} \tag{Nmh13}$$

$$p(r,t) = \bar{p} - Re\left\{\tilde{p}\frac{\sin \lambda r}{\lambda r}\;\frac{e^{i\omega t}}{\cos \lambda A_o}\right\} \tag{Nmh14}$$

The entire flow has thus been determined in terms of the prescribed quantities A_o, R_o, η, ω, and $\tilde{p}$.

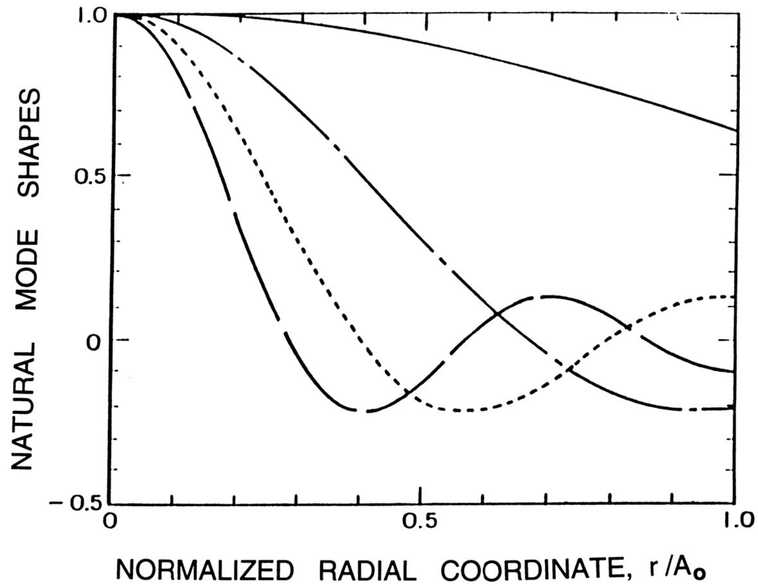

Figure 2: Natural mode shapes as a function of the normalized radial position, r/A_o, in the cloud for various orders $m = 1$ (solid line), 2 (dash-dotted line), 3 (dotted line), 4 (broken line). The arbitrary vertical scale represents the amplitude of the normalized undamped oscillations of the bubble radius, the pressure, and the bubble concentration per unit liquid volume. The oscillation of the velocity is proportional to the slope of these curves.

Note first that the cloud has a number of natural frequencies and modes of oscillation. From equation (Nmh10) it follows that, if $\tilde{p}$ were zero, oscillations would only occur if

$$\omega = \omega_n \quad \underline{\text{or}} \quad \lambda A_o = (2m-1)\frac{\pi}{2} \ , \ m = 0 \ , \ \pm 2 \ldots \tag{Nmh15}$$

and, therefore, using equation (Nmh11) for λ, the natural frequencies, ω_m, of the cloud are found to be:

1. $\omega_\infty = \omega_n$, the natural frequency of an individual bubble in an infinite liquid, and
2. $\omega_m = \omega_n \left[1 + 16\eta R_o A_o^2/\pi(2m-1)^2\right]^{\frac{1}{2}}$; $m = 1, 2, \ldots$, which is an infinite series of frequencies of which ω_1 is the lowest. The higher frequencies approach ω_n as m tends to infinity.

The lowest natural frequency, ω_1, can be written in terms of the mean void fraction, $\alpha_o = \eta v_o/(1 + \eta v_o)$, as

$$\omega_1 = \omega_n \left[1 + \frac{4}{3\pi^2}\,\frac{A_o^2}{R_o^2}\,\frac{\alpha_o}{1-\alpha_o}\right]^{-\frac{1}{2}} \tag{Nmh16}$$

Hence, the natural frequencies of the cloud will extend to frequencies much smaller than the individual bubble frequency, ω_n, if the initial void fraction, α_o, is much larger than the square of the ratio of bubble size to cloud size ($\alpha_o \gg R_o^2/A_o^2$). If the reverse is the case ($\alpha_o \ll R_o^2/A_o^2$), all the natural frequencies of the cloud are contained in a small range just below ω_n.

Typical natural modes of oscillation of the cloud are depicted in figure 2, where normalized amplitudes of the bubble radius and pressure fluctuations are shown as functions of position, r/A_o, within the cloud. The amplitude of the radial velocity oscillation is proportional to the slope of these curves. Since each bubble is supposed to react to a uniform far field pressure, the validity of the model is limited to wave numbers, m, such that $m \ll A_o/R_o$. Note that the first mode involves almost uniform oscillations of the bubbles at all radial positions within the cloud. Higher modes involve amplitudes of oscillation near the center of the cloud, that become larger and larger relative to the amplitudes in the rest of the cloud. In

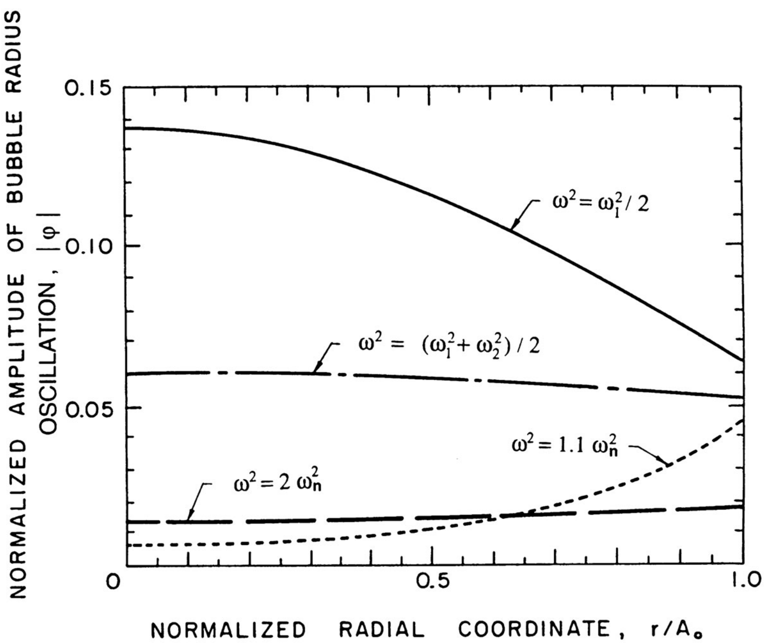

Figure 3: The distribution of bubble radius oscillation amplitudes, $|\varphi|$, within a cloud subjected to forced excitation at various frequencies, ω, as indicated (for the case of $\alpha_o(1-\alpha_o)A_o^2/R_o^2 = 0.822$). From d'Agostino and Brennen (1989).

effect, an outer shell of bubbles essentially shields the exterior fluid from the oscillations of the bubbles in the central core, with the result that the pressure oscillations in the exterior fluid are of smaller amplitude for the higher modes.