Acoustics of Bubbly Mixtures

One class of phenomena in which bubble dynamics can play an important role is the acoustics of bubble/liquid mixtures. When the acoustic excitation frequency approaches the natural frequency of the bubbles, the latter no longer respond in the quasistatic manner assumed in sections (Nl), and both the propagation speed and the acoustic attenuation are significantly altered. A review of this subject is given by van Wijngaarden (1972) and we will include here only a summary of the key results. This class of problems has the advantage that the magnitude of the perturbations is small so that the equations of the preceding section can be greatly simplified by linearization. Hence the pressure, p, will be represented by the following sum:

$$p = \bar{p} + Re\left\{\tilde{p}e^{i\omega t}\right\} \tag{Nmc1}$$

where $\bar{p}$ is the mean pressure, ω is the frequency, and $\tilde{p}$ is the small amplitude pressure perturbation. The response of a bubble will be similarly represented by a perturbation, φ, to its mean radius, R_o, such that

$$R = R_o\left[1 + Re\left\{\varphi e^{i\omega t}\right\}\right] \tag{Nmc2}$$

and the linearization will neglect all terms of order φ^2 or higher.

The literature on the acoustics of dilute bubbly mixtures contains two complementary analytical approaches. Foldy (1945) and Carstensen and Foldy (1947) applied the classical acoustical approach and treated the problem of multiple scattering by randomly distributed point scatterers representing the bubbles. The medium is assumed to be very dilute ($\alpha \ll 1$). The multiple scattering produces both coherent and incoherent contributions. The incoherent part is beyond the scope of this text. The coherent part, which can be represented by equation (Nmc1), was found to satisfy a wave equation and yields a dispersion relation for the wavenumber, κ, of plane waves, that implies a phase velocity, $c_\kappa = \omega/\kappa$, given by (see van Wijngaarden 1972)

$$\frac{1}{c_\kappa^2} = \frac{\kappa^2}{\omega^2} = \frac{1}{c_L^2} + \frac{1}{c_o^2}\left[1 - \frac{i\delta_d\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right]^{-1} \tag{Nmc3}$$

Here c_L is the sonic speed in the liquid, c_o is the sonic speed arising from equation (Nlb11) when $\alpha\rho_G \ll (1-\alpha)\rho_L$,

$$c_o^2 = k\bar{p}/\rho_L\alpha(1-\alpha) \tag{Nmc4}$$

ω_n is the natural frequency of a bubble in an infinite liquid (section (Ngj)), and δ_d is a dissipation coefficient that will be discussed shortly. It follows from equation (Nmc3) that scattering from the bubbles makes the wave propagation dispersive since c_κ is a function of the frequency, ω.

As described by van Wijngaarden (1972) an alternative approach is to linearize the fluid mechanical equations (Nmb1), (Nmb2), and (Nmb3), neglecting any terms of order φ^2 or higher. In the case of plane wave propagation in the direction x (velocity u) in a frame of reference relative to the mixture (so that the mean velocity is zero), the convective terms in the Lagrangian derivatives, D/Dt, are of order φ^2 and the three governing equations become

$$\frac{\partial u}{\partial x} = \frac{\eta}{(1+\eta v)}\frac{\partial v}{\partial t} \tag{Nmc5}$$

$$\rho_L\frac{\partial u}{\partial t} = -(1+\eta v)\frac{\partial p}{\partial x} \tag{Nmc6}$$

$$R\frac{\partial^2 R}{\partial t^2} + \frac{3}{2}\left(\frac{\partial R}{\partial t}\right)^2 = \frac{1}{\rho_L}\left[p_V + p_{Go}\left(\frac{R_o}{R}\right)^{3k} - p\right] - \frac{2S}{\rho_L R} - \frac{4\nu_L}{R}\frac{\partial R}{\partial t} \tag{Nmc7}$$

Assuming for simplicity that the liquid is incompressible (ρ_L = constant) and eliminating two of the three unknown functions from these relations, one obtains the following equation for any one of the three perturbation quantities ($Q = \varphi$, $\tilde{p}$, or $\tilde{u}$, the velocity perturbation):

$$3\alpha_o(1-\alpha_o)\frac{\partial^2 Q}{\partial t^2} = \left[\frac{3kp_{Go}}{\rho_L} - \frac{2S}{\rho_L R_o}\right]\frac{\partial^2 Q}{\partial x^2} + R_o^2\frac{\partial^4 Q}{\partial x^2 \partial t^2} + 4\nu_L\frac{\partial^3 Q}{\partial x^2 \partial t} \tag{Nmc8}$$

where α_o is the mean void fraction given by $\alpha_o = \eta v_o/(1+\eta v_o)$. This equation governing the acoustic perturbations is given by van Wijngaarden, though we have added the surface tension term. Since the mean state must be in equilibrium, the mean liquid pressure, $\bar{p}$, is related to p_{Go} by

$$\bar{p} = p_V + p_{Go} - \frac{2S}{R_o} \tag{Nmc9}$$

and hence the term in square brackets in equation (Nmc8) may be written in the alternate forms

$$\frac{3kp_{Go}}{\rho_L} - \frac{2S}{\rho_L R_o} = \frac{3k}{\rho_L}(\bar{p} - p_V) + \frac{2S}{\rho_L R_o}(3k-1) = R_o^2\omega_n^2 \tag{Nmc10}$$

This identifies ω_n, the natural frequency of a single bubble in an infinite liquid (see section (Ngj)).

Results for the propagation of a plane wave in the positive x direction are obtained by substituting $q = e^{-i\kappa x}$ in equation (Nmc8) to produce the following dispersion relation:

$$c_\kappa^2 = \frac{\omega^2}{\kappa^2} = \frac{\left[\frac{3k}{\rho_L}(\bar{p} - p_V) + \frac{2S}{\rho_L R_o}(3k-1)\right] + 4i\omega\nu_L - \omega^2 R_o^2}{3\alpha_o(1-\alpha_o)} \tag{Nmc11}$$

Note that at the low frequencies for which one would expect quasistatic bubble behavior ($\omega \ll \omega_n$) and in the absence of vapor ($p_V = 0$) and surface tension, this reduces to the sonic velocity given by equation (Nlb11) when $\rho_G\,\alpha \ll \rho_L(1-\alpha)$. Furthermore, equation (Nmc11) may be written as

$$c_\kappa^2 = \frac{\omega^2}{\kappa^2} = \frac{R_o^2\omega_n^2}{3\alpha_o(1-\alpha_o)}\left[1 + i\frac{\delta_d\omega}{\omega_n} - \frac{\omega^2}{\omega_n^2}\right] \tag{Nmc12}$$

where $\delta_d = 4\nu_L/\omega_n R_o^2$. For the incompressible liquid assumed here this is identical to equation (Nmc3) obtained using the Foldy multiple scattering approach (the difference in sign for the damping term results from using $i(\omega t - \kappa x)$ rather than $i(\kappa x - \omega t)$ and is inconsequential).

In the above derivation, the only damping mechanism that was explicitly included was that due to viscous effects on the radial motion of the bubbles. As Chapman and Plesset (1971) have shown, other damping mechanisms can affect the volume oscillations of the bubble; these include the damping due to temperature gradients caused by evaporation and condensation at the bubble surface and the radiation of acoustic energy due to compressibility of the liquid. However, Chapman and Plesset (1971) and others have demonstrated that, to a first approximation, all of these damping contributions can be included by defining an *effective* damping, δ_d, or, equivalently, an effective liquid viscosity, $\mu_e = \omega_n R_o^2\delta_d/4$.

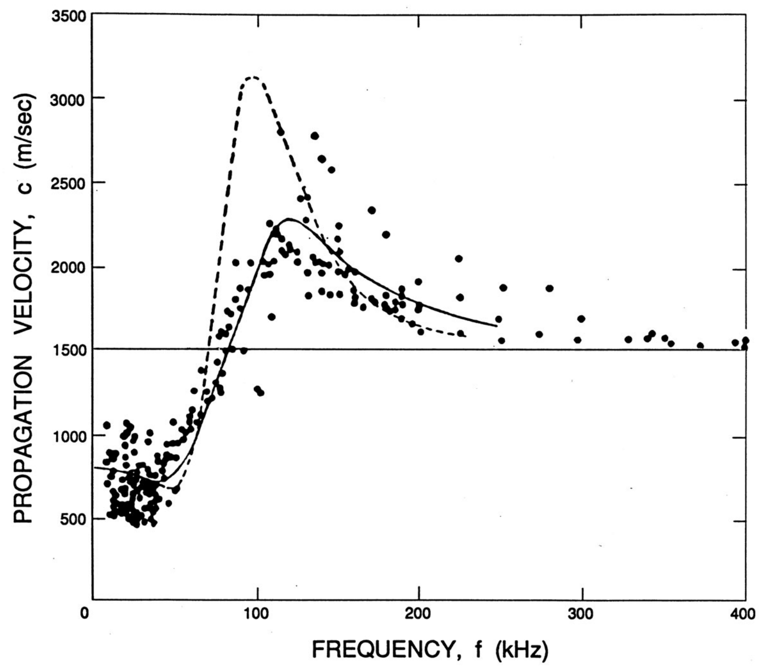

Figure 1: Sonic speed for water with air bubbles of mean radius, $R_o = 0.12mm$, and a void fraction, $\alpha = 0.0002$, plotted against frequency. The experimental data of Fox, Curley, and Larson (1955) is plotted along with the theoretical curve for a mixture with identical $R_o = 0.11mm$ bubbles (dotted line) and with the experimental distribution of sizes (solid line). These lines use $\delta_d = 0.5$.

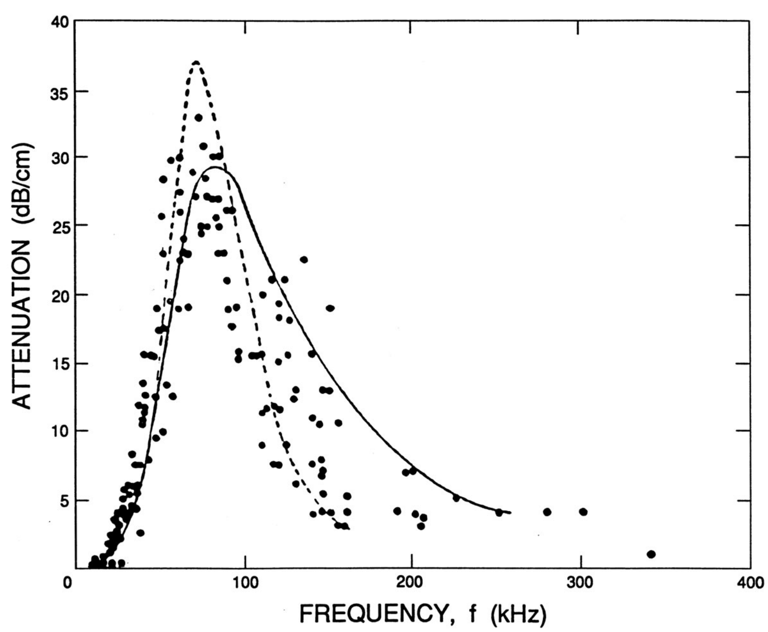

Figure 2: Values for the attenuation of sound waves corresponding to the sonic speed data of figure 1. The attenuation in dB/cm is given by 8.69 $Im\{\kappa\}$ where κ is in cm^{-1}.