Bubble Contents

In addition to the Rayleigh-Plesset equation, considerations of the bubble contents are necessary. To be fairly general, it is assumed that the bubble contains some quantity of non-condensable gas whose partial pressure is p_{Go} at some reference size, R_o, and temperature, T_∞. Then, if there is no appreciable mass transfer of gas to or from the liquid, it follows that

$$p_B(t) = p_V(T_B) + p_{Go}\left(\frac{T_B}{T_\infty}\right)\left(\frac{R_o}{R}\right)^3 \qquad \text{(Ngc1)}$$

In some cases this last assumption is not justified, and it is necessary to solve a mass transport problem for the liquid in a manner similar to that used for heat diffusion (see section (Ngi)).

It remains to determine $T_B(t)$. This is not always necessary since, under some conditions, the difference between the unknown T_B and the known T_∞ is negligible. But there are also circumstances in which the temperature difference, $(T_B(t) - T_\infty)$, is important and the effects caused by this difference dominate the bubble dynamics. Clearly the temperature difference, $(T_B(t) - T_\infty)$, leads to a different vapor pressure, $p_V(T_B)$, than would occur in the absence of such thermal effects, and this alters the growth or collapse rate of the bubble. It is therefore instructive to substitute equation (Ngc1) into (Ngb8) and thereby write the Rayleigh-Plesset equation in the following general form:

$$\underbrace{\frac{p_V(T_\infty) - p_\infty(t)}{\rho_L}}_{(1)} + \underbrace{\frac{p_V(T_B) - p_V(T_\infty)}{\rho_L}}_{(2)} + \underbrace{\frac{p_{Go}}{\rho_L}\left(\frac{T_B}{T_\infty}\right)\left(\frac{R_o}{R}\right)^3}_{(3)}$$
$$= R\frac{d^2R}{dt^2} + \underbrace{\frac{3}{2}\left(\frac{dR}{dt}\right)^2}_{(4)} + \underbrace{\frac{4\nu_L}{R}\frac{dR}{dt}}_{(5)} + \underbrace{\frac{2S}{\rho_L R}}_{(6)} \qquad \text{(Ngc2)}$$

The first term, (1), is the instantaneous tension or driving term determined by the conditions far from the bubble. The second term, (2), will be referred to as the *thermal term*, and it will be seen that very different bubble dynamics can be expected depending on the magnitude of this term. When the temperature difference is small, it is convenient to use a Taylor expansion in which only the first derivative is retained to evaluate

$$\frac{p_V(T_B) - p_V(T_\infty)}{\rho_L} = A(T_B - T_\infty) \qquad \text{(Ngc3)}$$

where the quantity A may be evaluated from

$$A = \frac{1}{\rho_L}\frac{dp_V}{dT} = \frac{\rho_V(T_\infty)\mathcal{L}(T_\infty)}{\rho_L T_\infty} \qquad \text{(Ngc4)}$$

using the Clausius-Clapeyron relation, $\mathcal{L}(T_\infty)$ being the latent heat of vaporization at the temperature T_∞. It is consistent with the Taylor expansion approximation to evaluate ρ_V and $\mathcal{L}$ at the known temperature T_∞. It follows that, for small temperature differences, term (2) in equation (Ngc2) is given by $A(T_B - T_\infty)$.

The degree to which the bubble temperature, T_B, departs from the remote liquid temperature, T_∞, can have a major effect on the bubble dynamics, and it is necessary to discuss how this departure might be evaluated. The determination of $(T_B - T_\infty)$ requires two steps. First, it requires the solution of the heat diffusion equation,

$$\frac{\partial T}{\partial t} + \frac{dR}{dt}\left(\frac{R}{r}\right)^2 \frac{\partial T}{\partial r} = \frac{\mathcal{D}_L}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) \tag{Ngc5}$$

to determine the temperature distribution, $T(r,t)$, within the liquid ($\mathcal{D}_L$ is the thermal diffusivity of the liquid). Second, it requires an energy balance for the bubble. The heat supplied to the interface from the liquid is

$$4\pi R^2 k_L \left(\frac{\partial T}{\partial r}\right)_{r=R} \tag{Ngc6}$$

where k_L is the thermal conductivity of the liquid. Assuming that all of this is used for vaporization of the liquid (this neglects the heat used for heating or cooling the existing bubble contents, which is negligible in many cases), one can evaluate the mass rate of production of vapor and relate it to the known rate of increase of the volume of the bubble. This yields

$$\frac{dR}{dt} = \frac{k_L}{\rho_V \mathcal{L}}\left(\frac{\partial T}{\partial r}\right)_{r=R} \tag{Ngc7}$$

where k_L, ρ_V, $\mathcal{L}$ should be evaluated at $T = T_B$. If, however, $T_B - T_\infty$ is small, it is consistent with the linear analysis described earlier to evaluate these properties at $T = T_\infty$.

The nature of the thermal effect problem is now clear. The thermal term in the Rayleigh-Plesset equation (Ngc2) requires a relation between $(T_B(t) - T_\infty)$ and $R(t)$. The energy balance equation (Ngc7) yields a relation between $(\partial T/\partial r)_{r=R}$ and $R(t)$. The final relation between $(\partial T/\partial r)_{r=R}$ and $(T_B(t) - T_\infty)$ requires the solution of the heat diffusion equation. It is this last step that causes considerable difficulty due to the evident nonlinearities in the heat diffusion equation; no exact analytic solution exists. However, the solution of Plesset and Zwick (1952) provides a useful approximation for many purposes. This solution is confined to cases in which the thickness of the thermal boundary layer, δ_T, surrounding the bubble is small compared with the radius of the bubble, a restriction that can be roughly represented by the identity

$$R \gg \delta_T \approx (T_\infty - T_B)/\left(\frac{\partial T}{\partial r}\right)_{r=R} \tag{Ngc8}$$

The Plesset-Zwick result is that

$$T_\infty - T_B(t) = \left(\frac{\mathcal{D}_L}{\pi}\right)^{\frac{1}{2}} \int_0^t \frac{[R(x)]^2 (\frac{\partial T}{\partial r})_{r=R(x)} dx}{\left\{\int_x^t [R(y)]^4 dy\right\}^{\frac{1}{2}}} \tag{Ngc9}$$

where x and y are dummy time variables. Using equation (Ngc7) this can be written as

$$T_\infty - T_B(t) = \frac{\mathcal{L}\rho_V}{\rho_L c_{PL} \mathcal{D}_L^{\frac{1}{2}}}\left(\frac{1}{\pi}\right)^{\frac{1}{2}} \int_0^t \frac{[R(x)]^2 \frac{dR}{dt} dx}{[\int_x^t R^4(y) dy]^{\frac{1}{2}}} \tag{Ngc10}$$

This can be directly substituted into the Rayleigh-Plesset equation to generate a complicated integro-differential equation for $R(t)$. However, for present purposes it is more instructive to confine our attention to regimes of bubble growth or collapse that can be approximated by the relation

$$R = R^* t^n \tag{Ngc11}$$

where R^* and n are constants. Then the equation (Ngc10) reduces to

$$T_\infty - T_B(t) = \frac{\mathcal{L}\rho_V}{\rho_L c_{PL} \mathcal{D}_L^{\frac{1}{2}}} R^* t^{n-\frac{1}{2}} C(n) \qquad \text{(Ngc12)}$$

where the constant

$$C(n) = n \left(\frac{4n+1}{\pi} \right)^{\frac{1}{2}} \int_0^1 \frac{z^{3n-1} dz}{(1 - z^{4n+1})^{\frac{1}{2}}} \qquad \text{(Ngc13)}$$

and is of order unity for most values of n of practical interest ($0 < n < 1$ in the case of bubble growth). Under these conditions the linearized form of the thermal term, (2), in the Rayleigh-Plesset equation (Ngc2) as given by equations (Ngc3) and (Ngc4) becomes

$$(T_B \;-\; T_\infty) \frac{\rho_V \mathcal{L}}{\rho_L T_\infty} = -\Sigma(T_\infty) C(n) R^* t^{n-\frac{1}{2}} \qquad \text{(Ngc14)}$$

where the thermodynamic parameter

$$\Sigma(T_\infty) \;=\; \frac{\mathcal{L}^2 \rho_V^2}{\rho_L^2 c_{PL} T_\infty \mathcal{D}_L^{\frac{1}{2}}} \qquad \text{(Ngc15)}$$

In section **??** it will be seen that this parameter, Σ, whose units are $m/sec^{\frac{3}{2}}$, is crucially important in determining the bubble dynamic behavior.