In the Absence of Thermal Effects; Bubble Growth

First we consider some of the characteristics of bubble dynamics in the absence of any significant thermal effects. This kind of bubble dynamic behavior is termed *inertially controlled* to distinguish it from the *thermally controlled* behavior discussed later. Under these circumstances the temperature in the liquid is assumed uniform and term (2) in the Rayleigh-Plesset equation (Ngc2) is zero.

For simplicity, it will be assumed that the behavior of the gas in the bubble is polytropic so that

$$p_G = p_{Go}\left(\frac{R_o}{R}\right)^{3k} \tag{Ngd1}$$

where k is approximately constant. Clearly $k = 1$ implies a constant bubble temperature and $k = \gamma$ would model adiabatic behavior. It should be understood that accurate evaluation of the behavior of the gas in the bubble requires the solution of the mass, momentum, and energy equations for the bubble contents combined with appropriate boundary conditions that will include a thermal boundary condition at the bubble wall.

With these assumptions the Rayleigh-Plesset equation becomes

$$\frac{p_V(T_\infty) - p_\infty(t)}{\rho_L} + \frac{p_{Go}}{\rho_L}\left(\frac{R_o}{R}\right)^{3k} = R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + \frac{4\nu_L}{R}\frac{dR}{dt} + \frac{2S}{\rho_L R} \tag{Ngd2}$$

Equation (Ngd2) without the viscous term was first derived and used by Noltingk and Neppiras (1950, 1951); the viscous term was investigated first by Poritsky (1952).

Equation (Ngd2) can be readily integrated numerically to find $R(t)$ given the input $p_\infty(t)$, the temperature T_∞, and the other constants. Initial conditions are also required and, in the context of cavitating flows, it is appropriate to assume that the bubble begins as a microbubble of radius R_o in equilibrium at $t = 0$ at a pressure $p_\infty(0)$ so that

$$p_{Go} = p_\infty(0) - p_V(T_\infty) + \frac{2S}{R_o} \tag{Ngd3}$$

and that $dR/dt|_{t=0} = 0$. A typical solution for equation (Ngd2) under these conditions is shown in figure 1; the bubble in this case experiences a pressure, $p_\infty(t)$, that first decreases below $p_\infty(0)$ and then recovers to its original value. The general features of this solution are characteristic of the response of a bubble as it passes through any low pressure region; they also reflect the strong nonlinearity of equation (Ngd2). The growth is fairly smooth and the maximum size occurs after the minimum pressure. The collapse process is quite different. The bubble collapses catastrophically, and this is followed by successive rebounds and collapses. In the absence of dissipation mechanisms such as viscosity these rebounds would continue indefinitely without attenuation.

Analytic solutions to equation (Ngd2) are limited to the case of a step function change in p_∞. Nevertheless, these solutions reveal some of the characteristics of more general pressure histories, $p_\infty(t)$, and are therefore valuable to document. With a constant value of $p_\infty(t > 0) = p^*_\infty$, equation (Ngd2) is integrated by multiplying through by $2R^2 dR/dt$ and forming time derivatives. Only the viscous term cannot be integrated in this way, and what follows is confined to the inviscid case. After integration, application of the initial condition $(dR/dt)_{t=0} = 0$ yields

$$\left(\frac{dR}{dt}\right)^2 = \frac{2(p_V - p^*_\infty)}{3\rho_L}\left\{1 - \frac{R_o^3}{R^3}\right\} + \frac{2p_{Go}}{3\rho_L(1-k)}\left\{\frac{R_o^{3k}}{R^{3k}} - \frac{R_o^3}{R^3}\right\} - \frac{2S}{\rho_L R}\left\{1 - \frac{R_o^2}{R^2}\right\} \tag{Ngd4}$$

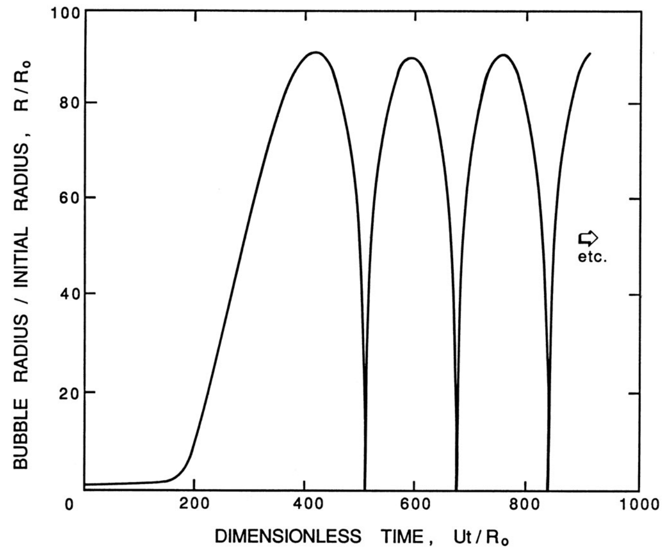

Figure 1: Typical solution of the Rayleigh-Plesset equation for a spherical bubble. The nucleus of radius, R_o, enters a low-pressure region at a dimensionless time of 0 and is convected back to the original pressure at a dimensionless time of 500. The low-pressure region is sinusoidal and symmetric about 250.

where, in the case of isothermal gas behavior, the term involving p_{Go} becomes

$$2\frac{p_{Go}}{\rho_L}\frac{R_o^3}{R^3}\ln\left(\frac{R_o}{R}\right) \quad \text{(Ngd5)}$$

By rearranging equation (Ngd4) it follows that

$$t = R_o \int_1^{R/R_o} \left\{ \frac{2(p_V - p_\infty^*)(1-x^{-3})}{3\rho_L} + \frac{2p_{Go}(x^{-3k} - x^{-3})}{3(1-k)\rho_L} - \frac{2S(1-x^{-2})}{\rho_L R_o x} \right\}^{-\frac{1}{2}} dx \quad \text{(Ngd6)}$$

where, in the case $k = 1$, the gas term is replaced by

$$\frac{2p_{Go}}{x^3}\ln x \quad \text{(Ngd7)}$$

This integral can be evaluated numerically to find $R(t)$, albeit indirectly.

Consider first the characteristic behavior for bubble growth that this solution exhibits when $p_\infty^* < p_\infty(0)$. Equation (Ngd4) shows that the asymptotic growth rate for $R \gg R_o$ is given by

$$\frac{dR}{dt} \rightarrow \left\{ \frac{2}{3}\,\frac{(p_V - p_\infty^*)}{\rho_L} \right\}^{\frac{1}{2}} \quad \text{(Ngd8)}$$

Thus, following an initial period of acceleration, the velocity of the interface is relatively constant. It should be emphasized that equation (Ngd8) implies explosive growth of the bubble, in which the volume displacement is increasing like t^3.

Now contrast the behavior of a bubble caused to collapse by an increase in p_∞ to p_∞^*. In this case when $R \ll R_o$ equation (Ngd4) yields

$$\frac{dR}{dt} \rightarrow -\left(\frac{R_o}{R}\right)^{\frac{3}{2}} \left\{ \frac{2(p_\infty^* - p_V)}{3\rho_L} + \frac{2S}{\rho_L R_o} - \frac{2p_{Go}}{3(k-1)\rho_L}\left(\frac{R_o}{R}\right)^{3(k-1)} \right\}^{\frac{1}{2}} \quad \text{(Ngd9)}$$

where, in the case of $k = 1$, the gas term is replaced by $2p_{Go}\ln(R_o/R)/\rho_L$. However, most bubble collapse motions become so rapid that the gas behavior is much closer to adiabatic than isothermal, and we will therefore assume $k \neq 1$.

For a bubble with a substantial gas content the asymptotic collapse velocity given by equation (Ngd9) will not be reached and the bubble will simply oscillate about a new, but smaller, equilibrium radius. On the other hand, when the bubble contains very little gas, the inward velocity will continually increase (like $R^{-3/2}$) until the last term within the curly brackets reaches a magnitude comparable with the other terms. The collapse velocity will then decrease and a minimum size given by

$$R_{min} = R_o \left\{ \frac{1}{(k-1)} \frac{p_{Go}}{(p_\infty^* - p_V + 3S/R_o)} \right\}^{\frac{1}{3(k-1)}} \qquad \text{(Ngd10)}$$

will be reached, following which the bubble will rebound. Note that, if p_{Go} is small, R_{min} could be very small indeed. The pressure and temperature of the gas in the bubble at the minimum radius are then given by p_m and T_m where

$$p_m = p_{Go} \left\{ (k-1)(p_\infty^* - p_V + 3S/R_o)/p_{Go} \right\}^{k/(k-1)} \qquad \text{(Ngd11)}$$

$$T_m = T_o \left\{ (k-1)(p_\infty^* - p_V + 3S/R_o)/p_{Go} \right\} \qquad \text{(Ngd12)}$$

We will comment later on the magnitudes of these temperatures and pressures (see sections (Nhc) and (Nhh)).

The case of zero gas content presents a special albeit somewhat hypothetical problem, since apparently the bubble will reach zero size and at that time have an infinite inward velocity. In the absence of both surface tension and gas content, Rayleigh (1917) was able to integrate equation (Ngd6) to obtain the time, t_{tc}, required for total collapse from $R = R_o$ to $R = 0$:

$$t_{tc} = 0.915 \left(\frac{\rho_L R_o^2}{p_\infty^* - p_V} \right)^{\frac{1}{2}} \qquad \text{(Ngd13)}$$

It is important at this point to emphasize that while the results for bubble growth are quite practical, the results for bubble collapse may be quite misleading. Apart from the neglect of thermal effects, the analysis was based on two other assumptions that may be violated during collapse. Later we shall see that the final stages of collapse may involve such high velocities (and pressures) that the assumption of liquid incompressibility is no longer appropriate. But, perhaps more important, it transpires (see section (Nhd)) that a collapsing bubble loses its spherical symmetry in ways that can have important engineering consequences.