Gas Convection and Diffusion

Many physiological flows in tubes are used to transport dissolved substances from one location to another in the body. At its source the substance is diffused into the flow by a difference in the concentration in the surrounding tissue and the concentration in the tubular flow. At the delivery location the process is reversed and the substance is diffused out of the tubular flow into the surrounding tissue. While these transport processes of diffusion and convection occur in many locations in the body it will be sufficient herein to demonstrate some of the key mechanics by referring to a specific example, namely the transport of dissolved gas to and from the lungs.

If the mass flow rate, $\dot{m}$, of the dissolved substance from the bulk fluid into the surrounding tissue (per unit time per unit length of the tube) is proportional to the difference between the concentration in the bulk of the fluid, c_b, and the concentration at the wall, c_w, then we may define a mass transfer coefficient, h', by

$$h\prime \;=\; \dot{m}/\pi D(c_b - c_w) \qquad \text{(Ebbb1)}$$

where D is the diameter of the circular tube. This mass transfer coefficient, $h\prime$, is conveniently non-dimensionalized by defining the Sherwood Number, Sh, as

$$Sh \;=\; h\prime D/\mathcal{D} \qquad \text{(Ebbb2)}$$

where $\mathcal{D}$ is the mass diffusivity of the gas in the fluid.

Parenthetically, we note that the above definitions, precisely parallel the following definitions for thermal diffusion namely the definition of a heat transfer coefficient, h, which connects the heat flux from the bulk fluid into the surrounding tissue (per unit time per unit length of the tube), $\mathcal{Q}$, to the temperature difference $(T_b - T_w)$ (where the T_b is the bulk temperature of the fluid and T_w is the temperature of the surrounding tissue):

$$h \;=\; \mathcal{Q}/(T_b - T_w) \qquad \text{(Ebbb3)}$$

The corresponding non-dimensionalization of h is known as the Nusselt number, Nu, given by

$$Nu \;=\; c_p \rho_L h D/\alpha_L \;=\; hD/k \qquad \text{(Ebbb4)}$$

where k and α_L are the thermal conductivity and thermal diffusivity of the fluid whose density and specific heat at constant pressure are ρ_L and c_p.

These two non-dimensional diffusion numbers, Sh and Nu, which respectively determine the diffusion of dissolved gas and heat into or out of the fluid are typically dependent not only on the properties of the fluid but also on the nature of the fluid flow. When the Reynolds number, Re, of the tube flow (recall $Re = VD/\nu$ where V is the bulk velocity of the flow and $/nu$ is the kinematic viscosity of the fluid) is less than 2000 the flow is laminar (which is the case for most biological flows) and the Nusselt number (and presumably the Sherwood number) is about 3.6. When Re is greater than about 4000 and the flow is turbulent, the Nu and Sh will depend on the the Reynolds number and on Pr or Sc, defined respectively as $Pr = \nu/alpha_L$ and $Sc = \nu/\mathcal{D}$. In the thermal case, the empirical result known as the Dittus-Boelter correlation (Greenberg and Toups 1970, Incropera and DeWitt 2007) is often used to determine Nu:

$$Nu \;=\; 0.023 Re^{0.8} Pr^{0.4} \qquad \text{(Ebbb5)}$$

and the analogous correlation for mass diffusion would be

$$Sh = 0.023 Re^{0.8} Sc^{0.4} \quad \text{(Ebbb6)}$$

As the flow proceeds through an organ like the lungs, the gas concentration difference $(c_b - c_w)$ changes with distance, x, along the tube. Consequently the net loss of dissolved gas through the wall over an incremental distance, dx, must be equal to the change in convected dissolved gas over that distance and therefore

$$-\dot{m}dx = \rho V A(c_b(x+dx) - c_b(x) - c_w(x+dx) + c_w(x)) \quad \text{(Ebbb7)}$$

where V is the bulk fluid velocity and $A = \pi D^2/4$ is the tube cross-sectional area It follows that

$$-\dot{m} = \rho V A \frac{d(c_b - c_w)}{dx} \quad \text{(Ebbb8)}$$

and substituting for $\dot{m}$ from equations (Ebbb1) and (Ebbb2) yields the evolution equation for $(c_b - c_w)$ as a function of x namely

$$\frac{d(c_b - c_w)}{dx} = -\frac{4\mathcal{D}\ Sh}{V\ D^2}(c_b - c_w) \quad \text{(Ebbb9)}$$

Assuming a constant Sherwood number (order $Sh = 3.6$) this leads to an exponential decline in concentration difference, $(c_b - c_w)$, of

$$(c_b - c_w) \propto \exp\{-\frac{4\ Sh\ \mathcal{D}}{V\ D^2}\ x\} \quad \text{(Ebbb10)}$$

and, therefore, to a typical half-distance, $x_{1/2}$, of

$$\frac{x_{1/2}}{D} = 0.693\ \frac{V\ D}{4\ Sh\ \mathcal{D}} = 0.173\ \frac{V\ D}{Sh\ \mathcal{D}} \quad \text{(Ebbb11)}$$

over which the concentration difference declines to half of its entrance value.

By way of a numerical example we note that the diameter of the smallest tubes in the microcirculation is $D = 10^{-5}m$, the typical velocity of blood in the microcirculation is $V = 10^{-2}m/s$ and the typical mass diffusivity of oxygen or carbon dioxide in water is about $\mathcal{D} = 2 \times 10^{-9}m^2/s$. If the Sherwood number has a value of order unity then it follows that the typical $x_{1/2}/D$ for the diffusion of the gases into or out of the microcirculation is of order 10 diameters. Thus the mass diffusion in the microcirculation is accomplished in a short length of the microcirculation tubes. On the other hand in arteries of diameter $D = 10^{-2}m$ and velocity of order $V = 10^{-1}m/s$ the typical $x_{1/2}/D$ for the diffusion of the gases is of order 10^5 and significant diffusion of gas would not occur.